Q1. (**True or False**) Please circle the correct answer. Each question worths 0.5 points.

(i) The subset $W := \{(x,y) \in \mathbb{R}^2 : x^2 = y^2\}$ is a subspace of the vector space $\mathbb{R}^2$ over $\mathbb{R}$.

$(1,-1), (1,1) \in W$
but $(1,-1)+(1,1) \notin W$.

TRUE FALSE

(ii) Let $S = \{v_1, v_2, v_3\}$ be a subset of $\mathbb{R}^3$ where none of them is a scalar multiple of another. Then S must be linearly independent.

Consider $\{(1,0,0),(0,1,0),(1,1,0)\}$

TRUE FALSE

(iii) The subspace of $n \times n$ symmetric matrices $\{A \in M_{n\times n}(\mathbb{R}) : A^t = A\}$ has dimension $n(n+1)/2$ as a vector space over $\mathbb{R}$.

$(E^{ij})_{kl} = \begin{cases} 1, & (k,l)=(i,j) \\ 0, & \text{otherwise} \end{cases}$, $\{E^{ij} : 1 \le i \le j \le n\}$ is a basis.

TRUE FALSE

(iv) Let $A, B \in M_{n\times n}(\mathbb{R})$. If AB is invertible, then both A and B must be invertible.

See Section 2.4 Q9

TRUE FALSE

(v) Any system of 3 linear equations in 5 unknowns has at least one solution.

Consider $\begin{cases} x_1+x_2+x_3+x_4+x_5=0 \\ x_1+x_2=0 \\ x_1+x_2=1 \end{cases}$

TRUE FALSE

(vi) If $T : V \to W$ is a linear transformation where V is a finite dimensional vector space over $\mathbb{F}$, then nullity(T)+ rank(T) =dim(V).

Dimension Theorem.

TRUE FALSE

(vii) Given $x_1, x_2 \in V$ and $y_1, y_2 \in W$, there exists a linear transformation $T : V \to W$ such that $T(x_1) = y_1$ and $T(x_2) = y_2$.

Consider $V = W = \mathbb{R}^2$
$x_1 = (1,0)$ $y_1 = (1,0)$
$x_2 = (2,0)$ $y_2 = (1,1)$

TRUE FALSE

(viii) There exists an $A \in M_{n\times n}(\mathbb{R})$ with no eigenvectors (over $\mathbb{R}$).

Consider $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

TRUE FALSE

Q2. Consider the 2×2 real matrix $A = \begin{pmatrix} -1 & -6 \\ 2 & 6 \end{pmatrix}$,

(i) (2 points) Find all the eigenvalues of A (over $\mathbb{R}$).

The characteristic polynomial of A is

$$\det \begin{pmatrix} -1-\lambda & -6 \\ 2 & 6-\lambda \end{pmatrix} = \lambda^2 - 5\lambda + 6 = (\lambda-2)(\lambda-3).$$

Hence the eigenvalues of A are 2 and 3.

(ii) (2 points) For each eigenvalue λ found in (i), find <u>*all*</u> the eigenvectors with eigenvalue λ.

For $\lambda = 2$, $(A-2I)x = 0$

$$\Leftrightarrow \begin{pmatrix} -3 & -6 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow x_1 + 2x_2 = 0.$$

So, all the eigenvectors w.r.t. $\lambda=2$ are $t\begin{pmatrix} 2 \\ -1 \end{pmatrix}$, $t \in \mathbb{R} \setminus \{0\}$

For $\lambda = 3$, $(A-3I)x = 0$

$$\Leftrightarrow \begin{pmatrix} -4 & -6 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow 2x_1 + 3x_2 = 0$$

So, all the eigenvectors w.r.t. $\lambda=3$ are $t\begin{pmatrix} 3 \\ -2 \end{pmatrix}$, $t \in \mathbb{R} \setminus \{0\}$.

(continued)

(iii) (2 points) Find an invertible 2×2 matrix Q such that $Q^{-1}AQ$ is diagonal.

Let $Q = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}$. Then $Q^{-1} = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}$ and

$$Q^{-1} A Q = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

—END OF QUIZ 1—